Interactive Visual Report 4 - MCEN 5151

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1 A Photo of the Flow

Figure 1: A frame from the video

2 Background... Poem?

Atop the shaketable, dulcet tones sit -

Shaken gently, forming peaks and pits.

Vibratory modes resonate through and through

i.e., there's pretty lumps in this goo.

What does it say about us dull-witted apes,

that we are so encapsulated by elementary shapes?

Oh, to be the punchline of such divine jest!

And now, to ignore these pressing thoughts and explore the rest.

3 CoolMath4kids

Standing waves of a fluid can be described using partial differential equations (PDEs). The world of PDEs is vast, and we know how to solve about 10 of them. Conveniently, I have chosen an experiment where the physics of the flow are described the polar 2d wave equation. There is some setup required for the equation.

Consider an open disk α. Physically, α is the perimeter of the paint, with radius a. Let $\Omega\alpha$ be the boundary of the paint. Note that the disk is centered at the origin. Next, for any point (x, y) in the paint, at a given time t let the height be denoted as $g(x, y, t)$. With these parameters defined, the wave equation takes the form:

$$\frac{\partial^2 g}{\partial t^2} = c^2 \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) \text{ for } (x, y) \in \mathbb{R} \text{ where: } g = 0 \text{ on } \Omega\alpha \tag{1}$$

This equation is the standard form in Cartesian coordinates. Because our disk exists within a circular geometry, it is likely convenient to use polar coordinates. Following a disgusting conversion, which I will not perform here, the below equation is produced:

$$\frac{\partial^2 g}{\partial t^2} = c^2 \left(\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} \right) \text{ for } |r| < a, |\theta| \leq 2\pi, \text{ where: } g = 0 \text{ for } r = a \tag{2}$$

c is a positive constant which is dependant on physical parameters, it describes the transverse wave speed of propagation. For now, this value will be left a constant, and I will use the model to estimate the form of the fluid, rather than specific dimensions.

Now, we are going to look at the axisymmetric case of the equation, where g does not change with θ. This assumption holds for the first seven seconds of the video, while the fluid is speeding up to the frequency of 17.5 Hz, the set oscillation of the table. I will derive the solution to this case and not the general, because it is easier. The axisymmetric equation looks like:

$$\frac{\partial^2 g}{\partial t^2} = c^2 \left(\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} \right) \text{ for } |r| < a, |\theta| \leq 2\pi \text{ where: } g = 0 \text{ for } r = a \tag{3}$$

Notice this is the same as the prior equation, but the last term in the spatail component is removed. To find solutions, separation of variables will be used. This technique assumes the solution is a product of two functions of a single variable. This can then be substituted into the original PDE, separated, and integrated in order to find solutions. Thus it is assumed:

$$g(r,t) = R(r)T(t) \tag{4}$$

When substituted into the PDE, and each variable is moved to one side, the equation yields:

$$\frac{T''(t)}{c^2 T(t)} = \frac{1}{R(r)} \left(R''(r) + \frac{1}{r} R'(r) \right) \tag{5}$$

c is moved to the temporal side of the equation for mathematical convenience. Now, this is a difficult step to take conceptually, but stick with me here. The left hand side of the equation solely depends on t, and the right hand side solely depends on r, thus it follows that both sides must be equal to some constant K (if you are having difficulty understanding this, consider what would happen if you were to take the derivative of the whole equation with respect to t or r. Either way, multivariable calculus tells us that one half of the equation would equal zero; and what is the derivative of a constant? Zero!). We can thus write:

$$\frac{T''(t)}{c^2 T(t)} = \frac{1}{R(r)} \left(R''(r) + \frac{1}{r} R'(r) \right) = K \in \mathbb{R} \tag{6}$$

Now, each equation can be individually related to K, forming a system of two ODE's (how cool is that?).

$$\frac{T''(t)}{c^2 T(t)} = K \tag{7}$$

$$\frac{1}{R(r)} \left(R''(r) + \frac{1}{r} R'(r) \right) = K \tag{8}$$

Now separating into standar form for ODE's gives:

$$T''(t) = Kc^2 T(t) \tag{9}$$

$$rR''(r) + R'(r) - KrR(r) = 0 \tag{10}$$

Ouch. It has been a long time since I derived this equation in Fourier Analysis. For some reason I can't quite remember, we say $K = -\lambda^2$. If I recall correctly, the value of K influences what solutions the temporal differential equation will generate. If $K > 0$ then $T(t)$ will generate exponential solutions, and if If $K < 0$ then $T(t)$ will generate sinusoidal solutions as a result of Euler's Formula which states that sin and cos are sums of complex exponential functions. It is not expected for our solution to diverge exponentially, thus $K < 0$ is chosen. Now each ODE can be solved, first the temporal solution:

$$T(t) = A\cos{(c\lambda t)} + B\sin{(c\lambda t)} \tag{11}$$

This can be solved with the method of undetermined coefficients, or via variation of parameters. Though it is important to note that the temporal equation was a standard form of ODE which generates linear combinations of sinusoids. Next, the spatial portion of the equation will be solved. In this equation, it can be seen that the ODE is a special case of a 0th order Bessel Function, which are solutions to Bessels Differential Equation. These solutions are not fun to work with, and are commonly described as $J_n(r)$ where n denotes the order. The curves of varying degree in 2 spatial dimensions look like:

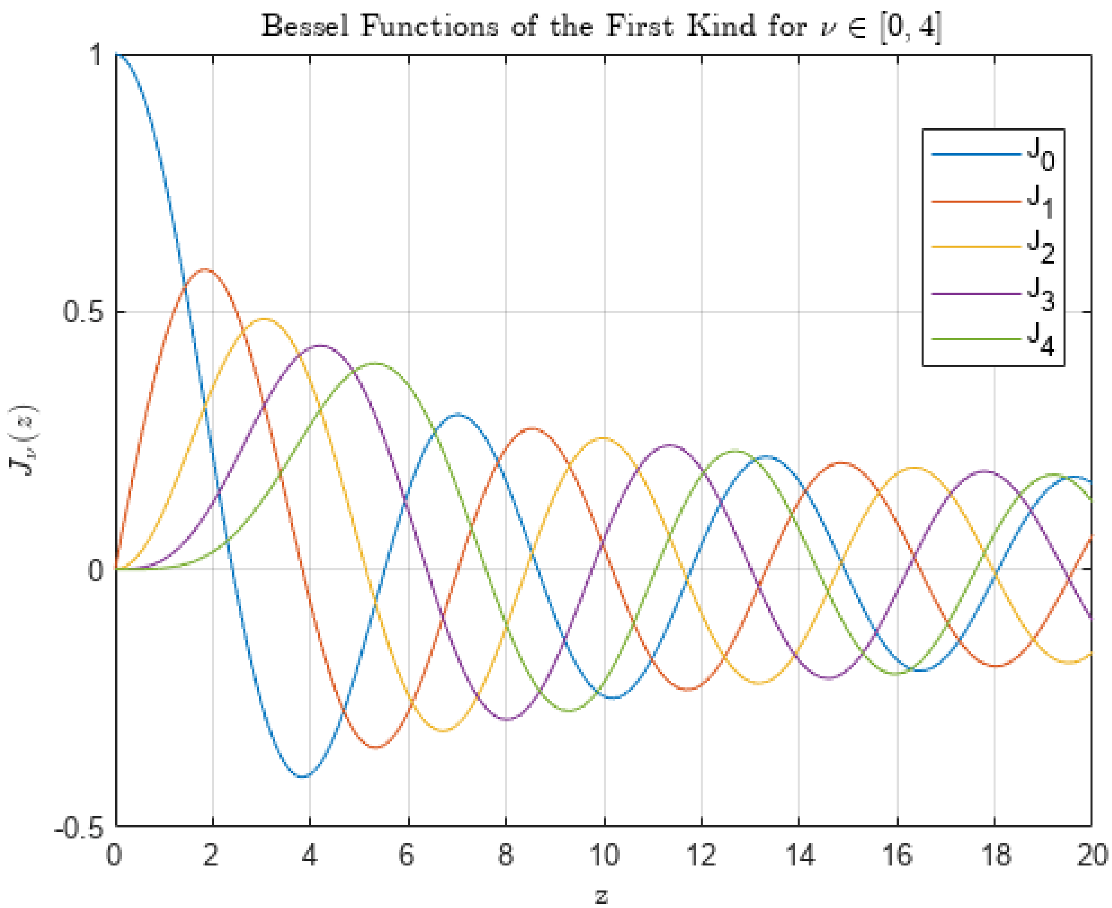

Figure 2: Bessel Functions of Varying Degrees

$J_0(x)$ is actually a commonly known function in one-dimensional Cartesian coordinates:

$$J_0(x) = \frac{\sin{(x)}}{x} \tag{12}$$

Which can be seen by the blue curve above. These functions are non-orthogonal to each other under the normal function inner-product space, as the frequency of their roots is not a constant value, and instead changes with x. There is not enough time for this, we must move on.

After our brief foray into Bessel Functions, we can now construct the solution for the spatial equation, or $R(r)$.

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \tag{13}$$

Where Y_0 is another Bessel Function, but it is unbounded and thus infinite at $r = 0$. Because this situation is non-causal, c_2 is assumed to be zero. It is assumed that c_1 is absorbed into the coefficients of the temporal equation. Now we must apply our boundary condition from the beginning,

$$R(a) = 0 = J_0(\lambda a) \tag{14}$$

As can be seen by the graph above, the Bessel Function has an infinite number of positive roots, which we will call σ_n. Thus we see $\lambda a = \sigma_n$ for $n = 1, 2, ...,$

And finally, we arrive at our solution for the spatail equation:

$$R(r) = J_0\left(\frac{\sigma_n}{a} r\right) \tag{15}$$

And, as assumed, in the beginning of this section:

$$g(r,t) = R(r)T(t) \tag{16}$$

Or

$$\boxed{g_n(r,t) = \left(A\cos\left(c\lambda_n t\right) + B\sin\left(c\lambda_n t\right)\right) J_0\left(\lambda_n r\right) \text{ for } n = 1, 2, ..., \text{ where } \lambda_n = \frac{\sigma_n}{a}} \tag{17}$$

This expression can now analytically describe the height of the height of any point of the liquid at any time! Here is an image that shows the graph of the solution at a few different values of n:

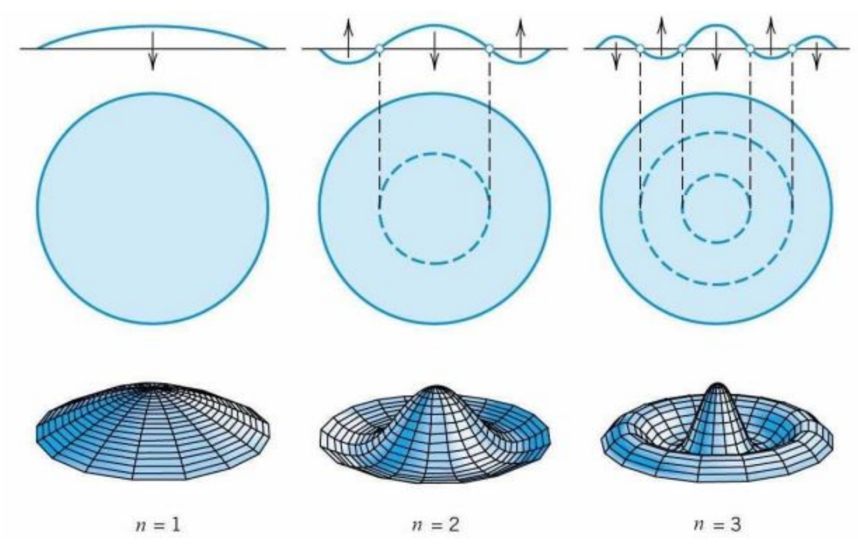

Figure 3: Axisymmetric case for increasing values of n

I scrolled through the video to see if I could find any points that closely matched one of the harmonics, and came up with something pretty neat:

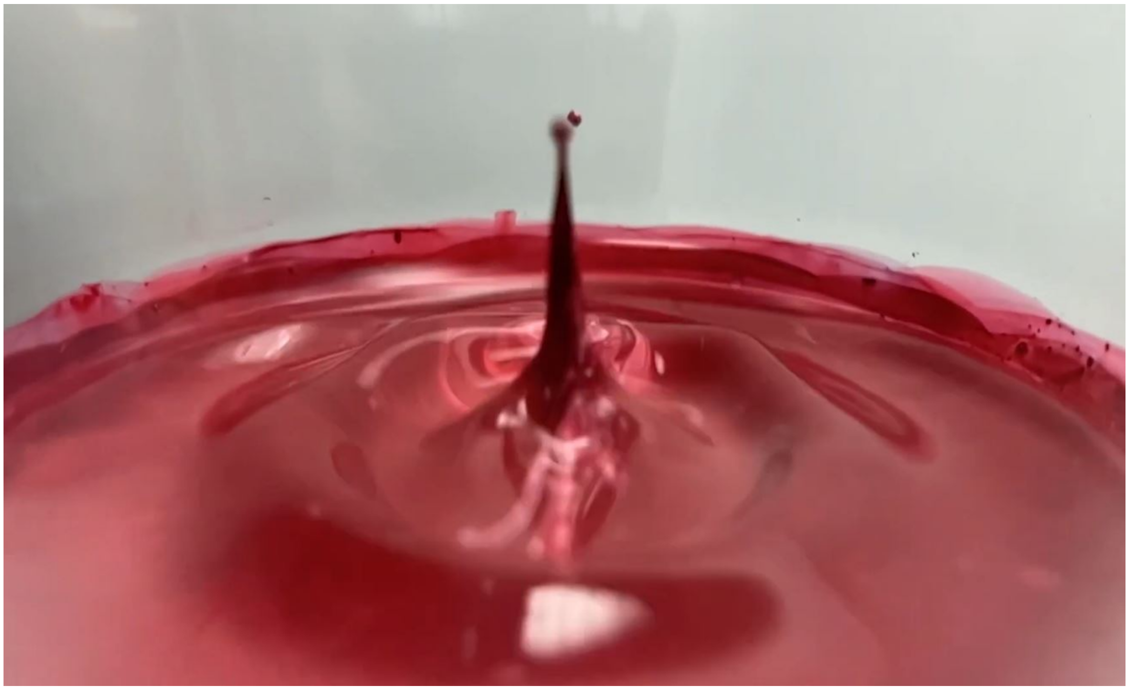

Figure 4: 3rd Mode Harmonic in Vibration of Fluid

All things considered, this looks shockingly similar to the $n = 3$ case of the solution! This frame was taken at about 6 seconds into the video, after which the fluid begins to oscillate closer to the 17.5 Hz frequency supplied by the plate. This caused larger values of n to be introduced, as well as the $\partial\theta$ term to be reintroduced to the spatial side of the equation. The middle actually shoots upwards too, due to the existence of the previously discussed Y_0 Bessel Function. Unlike a solid membrane, the fluid can detach and shoot upwards; under ideal conditions, this would happen infinitely at one point. But, given the unevenness of the vibrations, viscosity, and surface tension, the fluid only gets shot upwards some small amount. I could talk about this stuff all day, but alas there are more fish to fry.

4 Experiment Setup

There are a small number of materials that are necessary for this experiment, the most notable of which is a shake table. There are multiple ways once could go about accessing one. I chose to pay exorbitant amounts of tuition money which guarantees me limited access to basic engineering tools. Other options include stealing one from a lab, or purchasing one for ludicrous amounts of money. Once you have gained access to a shake-table, regardless of morality, all that is still needed is a Tupperware, some acrylic paint, and some water. Depending on the shape of the dish, different waves and vibratory modes will form. A cylindrical vessel can generate a number of different modes dependant on frequency

First, mix the paint (color being your choice, of course) with water until the viscosity is close to that of water. Pour about half of one inch of paint into the Tupperware, and be sure to pour into the center. Any paint that splashes on the side should be cleaned up with a paper towel. From here, affix the Tupperware to the shake table, potentially at the top (as it would be less useful to the experiment if it was put on a part of the table that doesn't vibrate). Begin to vibrate the table somewhere within the range of 10-25Hz, and film the results. Before vibration, you can add drops of different color paint to watch them diffuse into the larger body. Frankly, it is up to you. What do I care? I'm just a disembodied voice.

5 Photographic Technique

This was filmed in slow motion on my iPhone 13 Pro, at 240 frames per second. Thus, when shown at 60 fps, this video is 4x slower than normal speed. It was shot at a resolution of 1920 by 1080 pixels. Other than that, I held the phone up to the side of the glass and pressed record. In post processing, I added a title and credits to the video, and played Chopin's Opus 9 Number 2. A fairly pedestrian choice, I know, but I wanted something to please the masses.

6 Intended Image Ideals, and Inevitable Shortcomings

I think this video came out well, for the most part. I think the art would have benefited from some way to mount the camera, as well as a better way to view the fluid. While the curvature of the glass did create interesting flow phenomena, it did make the boundary of the screen look fairly foggy. I am pleased that you can see the different vibratory modes form within the fluid itself. This video provided a wonderful opportunity to explore the PDEs I had only previously learned theoretically. I hope I was able to teach you something along the way as well. Captain Punctual out.